Two-Dimensional Analytical Solution for Tide-Induced Watertable Fluctuations in a Sandy Rhythmic Coastline

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ABSTRACT


Modelling of tide-induced watertable fluctuations in coastal unconfined aquifers has, for many years, been based on the assumptions of one-dimensional shallow flow and a vertical beach face. This approximation is based on the Boussinesq equation, neglecting variations in the coastline and beach slope. Here, a closed-form analytical solution for a two-dimensional unconfined coastal aquifer, taking account the sloping and variable coastline, is derived. In the appropriate limits, this new solution reduces to the vertical beach, one-dimensional solution. The effects of different beach slopes and the coastline variability for a sandy beach environment are investigated. The results indicate that both the coastline shape and beachface slope are important parameters affecting the solution for tide-driven coastal groundwater fluctuations.

ADDITIONAL INDEX WORDS: Unconfined aquifers, groundwater fluctuations, Boussinesq equation.

INTRODUCTION

Tide-induced watertable fluctuations in a sandy beach affect erosion, saltwater intrusion, contamination from groundwater discharge and biological activity. The fluctuations decay landward from the shoreline, and could have average levels higher than the mean sea level (KNIGHT, 1982).

Most previous analytical solutions have been limited to one-dimensional forms assuming a straight coastline. Two-dimensional approaches to coastal aquifers, which consider alongshore coastline variability, have only recently been investigated. Among these, SUN (1997) developed an analytical model where the tidal fluctuation in the aquifer is adjacent to an estuary. However, the boundary condition in this model ignored the effects of oceanic tides propagating and attenuating in the estuary. The model of LI et al. (2000a) took this into account, and developed a new analytical solution based on a Green's function approach. Their solution demonstrated that interactions between estuarine and oceanic tides on watertable fluctuations could be significant.

LI et al. (2001) later derived an approximation that took into account the influence of the coastline shape on coastal watertable fluctuations. This simplified approximation showed that often the straight-line L-shaped approximation is unrealistic. Recently, LI et al. (2002) developed an analytical solution for tide-induced watertable fluctuations in a coastal aquifer bounded by a periodic shoreline: they considered both sinusoidal and natural coastlines. However, their solution was only valid for the conventional Boussinesq equation. A more realistic, higher-order solution is required to get further insight into watertable fluctuations on natural beaches.

We recall that the Boussinesq equation results from the shallow flow approximation to Laplace's equation with a free-surface boundary condition (see below). In their analysis of this expansion, PARLANGE et al. (1984) pointed out that the second-order linearised solution to the shallow flow approximation was adequate to describe the groundwater free surface elevation when the amplitude of the motion is less than the time-averaged groundwater depth. They concluded that the inclusion of the second-order free surface flow gives a more accurate method to predict the water level compared with the first-order solution.

We now consider the Boussinesq model, which is the first-order term in the shallow-flow expansion. NIELSEN (1990) was the first to derive an analytical solution for a sloping beach, where the assumption of a fixed location of the shoreline boundary condition is relaxed. This solution produces only an approximation to the boundary condition at the intersection of the beach and the ocean. With the inclusion of beach slope component in the perturbation parameter, the NIELSEN (1990) solution is limited to large beach slopes.

LI et al. (2000b) modified the approach of NIELSEN (1990) to include the concept of a moving boundary. Their method overcame the inconsistent boundary condition imposed by NIELSEN (1990). However, like NIELSEN (1990), LI et al. (2000b) adopted the beach slope as the perturbation parameter, which limits their solution to large beach slopes.

The shortcoming of using the beach slope as the perturbation parameter was overcome by TEO et al. (2003 a, b), who developed an alternative perturbation scheme. However, all previous investigations for sloping beaches have been limited to one-dimensional cases.

In this paper, an analytical solution satisfying tide-induced watertable fluctuations for a periodically varying sandy coastline and a sloping beach is derived. The solution is based on the higher-order governing equation derived from the shallow flow expansion, in addition to the Boussinesq equation. With the new, two-dimensional solution, we investigate the effects of beach slope, higher-order components and coastline variation.

Boundary Value Problem

The phenomenon of ocean tides incident at a sloping beach is depicted in Figure 1. The horizontal x-axis extends positive inland from a fixed origin at the mean tidal level (MTL). The location of the intersection of the sloping beach boundary and the variable mean tidal level is defined by

\[ x_0(t) = A \cot \theta \cos \omega t \]  

(1)

where \( x_0(t) \) and \( A \) and \( \omega \) represent the beach slope, tidal amplitude and tidal frequency, respectively. At the interface of ocean and land at \( x_0 \), the initial watertable height can be defined as;

\[ h(x_0(t), y, t) = D(1 + A \cos \omega t) \]  

(2)
In equation (2), \( \alpha (=A/D) \) is a dimensionless mean tidal amplitude parameter, representing the ratio of tidal amplitude, \( A \), to the mean tidal height, \( D \). For an incompressible and inviscid fluid, the potential head \( (x,y,z,t) = z + p/g \), will satisfy the continuity equation which leads to the Laplace’s equation (Bear, 1972).

\[
\nabla^2 \phi = 0, \quad 0 \leq z \leq h(x,y,t) \tag{3}
\]

Equation (3) is to be solved subject to (1) and the following boundary conditions:

(i) Kinematic condition at the free surface:

\[
\eta \phi_z = K (\phi_x^2 + \phi_y^2 + \phi_z^2) - K \phi_z, \quad z = h \tag{4a}
\]

(ii) Bottom boundary condition:

\[
\phi_z = 0, \quad z = 0 \tag{4b}
\]

(iii) Upper free surface boundary condition:

\[
\phi = 0, \quad z = h \tag{4c}
\]

(iv) Far boundary condition:

\[
\phi_x = 0, \quad x = \infty \tag{4d}
\]

The soil properties in the equations are defined as \( n \) and \( K \) which are soil porosity and hydraulic conductivity, respectively. Equation (4a) describes the dynamic condition of the free surface oscillation. Equation (4b) states that vertical flux through the bottom boundary is zero. Equation (4c) sets the pressure head at the upper free surface to zero (i.e., atmospheric pressure) while Equation (4d) recognises that the tidal influence in the \( x \)-direction eventually becomes negligible. This condition also implies that regional flow in the aquifer is not considered. A sinusoidal coastline is considered in Figure 1(b). The coastline can be described by

\[
x_s = A_s \sin(k_s y_s) \tag{5}
\]

where \( A_s \) is the amplitude of the shoreline oscillation, \( k_s = 2\pi/L_s \) is the wave number of the shoreline oscillation while \( L_s \) is the wavelength.

### Non-Dimensional Parameters

To simplify the governing equation and the boundary conditions, the horizontal and vertical variables are non-dimensionalised by the tidal decay length \( (L) \) and the mean tidal height \( (D) \) respectively. The non-dimensional parameters are summarised below:

\[
X = \frac{x}{L}, \quad Y = \frac{y}{L}, \quad \beta = \frac{\phi}{L}, \quad \alpha = \frac{\phi}{D}, \quad T = \frac{t}{T}, \quad \varepsilon = \frac{D}{L} \tag{6}
\]

where \( \lambda \) is the wave number of the shoreline times the linear decay length and is defined as a shallow water perturbation parameter representing the ratio of the mean tidal height to the linear decay length. This parameter is entirely controlled by the material constants and the prescribed boundary condition.

### Perturbation Parameters

There are three independent parameters defined by the material and the boundary conditions: the shallow water parameter \( (\varepsilon) \), the mean tidal amplitude parameter \( (\alpha) \) and the coastline parameter \( (\beta) \). The solution constructed for the problem through a perturbation expansion should be valid for small, \( \varepsilon \), and \( \alpha \) and a wide range of beach slopes, \( \theta (0 < \theta = \pi/2) \).

The hydraulic conductivity \( K \) in general varies from 20 to 1000 m/d for fine sand to gravel (Bear, 1972). The tidal wave frequency is 2 cycles/d, \( \omega = 4 \pi \) while the mean tidal height \( (D) \) ranges from 1 to 10 m. For a sandy beach, will range from approximately 0.1 to 0.6. The tidal amplitude, \( A \), commonly is smaller than the mean tidal height, \( D \), which gives the mean tidal amplitude parameter \( \alpha < 1 \). Since \( \varepsilon \) and \( \alpha \) are both less than unity for sandy beaches, they can be adopted as perturbation parameters.

The potential head and the total watertable fluctuation both are perturbed in terms of the (small) parameters \( \varepsilon, \alpha, \beta \): 

\[
H \approx 1 + \sum_{m=0}^{l} \sum_{m=1}^{2} \sum_{l=0}^{m} \varepsilon^m \alpha^n \beta^l H_{nnn}, \quad \Phi \approx \sum_{m=0}^{l} \sum_{m=1}^{2} \sum_{l=0}^{m} \varepsilon^m \alpha^n \beta^l \phi_{nnn} \tag{7a,b}
\]

where the upper summation limits give 1-, 2-, and 4th order expansions for the shallow water \( (\varepsilon) \), the mean tidal amplitude \( (\alpha) \) and the coastline \( (\beta) \) parameters, respectively.

By introducing the perturbation expansion from Equation (7) into the governing equation (3) and the boundary conditions (4a) to (4d), the following linearised governing equations are obtained:

\[
O(\varepsilon^2 \alpha^2): 2H_{000} = \lambda^2 H_{011} + H_{010} \tag{8a}
\]
\[
O(\varepsilon^2 \alpha^2): 2H_{001} = \lambda^2 H_{010} + H_{011} \tag{8b}
\]
\[
+ \lambda \sin Y H_{010} - 2\lambda \cos Y H_{011} \tag{8b}
\]
\[
O(\varepsilon^2 \alpha^2): 2H_{012} = \lambda^2 H_{021} + H_{012} \tag{8c}
\]
\[
+ \frac{1}{2} \left[ 1 + \cos(2Y) \right] H_{010} \tag{8c}
\]

\[
O(\varepsilon^2 \alpha^2): 2H_{020} = \lambda^2 H_{020} + H_{020} \tag{9a}
\]
\[
+ \frac{1}{2} \left[ 1 + \cos(2Y) \right] H_{010} \tag{9a}
\]
Equations (13a) and (15a) are identical to those given by Barry et al. (1996) for the shallow flow approximation (where the capillary effect is neglected, as is the case here). The first and second-order tidal oscillations solved for the Boussinesq equation are identical to Jeng et al. (2003).

It has been reported by Jeng et al. (2003) that only minor differences of watertable fluctuations in a sandy beach are found in between orders of $O(\alpha^2)$ and $O(\alpha^4)$. Therefore, the perturbation solution solved up to $O(\alpha^4)$ is considered to be adequate.

First-Order Shallow Water Expansion: $O(\alpha^4)$

The governing equations (10a) and (10b) for the first-order oscillations are identical to governing equations (8a) and (8b), but now equations (12) imply that the solutions are zero,

$$H_{110} = 0 \quad \text{and} \quad H_{111} = 0$$

Due to the sloping beach components in the second-order oscillations, equations (11a) and (11b) are different from their vertical-slope equivalents. The solutions derived for equations (11a) and (11b) are, respectively,

$$O(\alpha^2): H_{120} = \frac{1}{2} \cot \theta \left[ (i+1) e^{-4X} - e^{4X} \right]$$

$$O(\alpha^2): H_{121} = \cot \theta \sin Y \left[ \frac{\lambda}{k_{011}} (e^{-4X} - e^{4X}) \right] + \frac{k_{000} k_{010} e^{2iX} (e^{-4X} - e^{4X})}{2 \sqrt{\lambda}}$$

Equations (13a)-(17b) provide the total solution for watertable fluctuations in higher-order form. The solutions derived for different orders can be assembled as a total watertable height in the following manner;

$$H = 1 + \alpha \left( H_{000} + \beta H_{011} + \beta^2 H_{110} \right) + \alpha^2 \left( H_{020} + \beta H_{021} + \beta \alpha \left( H_{120} + \beta H_{121} \right) \right)$$

Equation (18) is a new expression for modelling flow in a two-dimensional unconfined coastal aquifer. Teo et al. (2003a,b) derived the corresponding one-dimensional solution. The new solution in equation (18) is identical to Teo et al. (2003a,b) when $\beta = 0$.

The solution in equation (18) can be further reduced to vertical beach case by setting $\theta = \pi \alpha$. Then, the solution is identical to that given by Jeng et al. (2003).

RESULTS AND DISCUSSION

Watertable Fluctuations for a Sandy Beach in a Temporal Domain

The primary aim of the paper is to investigate the effect of the higher-order components, beach slopes and the coastline variability on the fluctuations. A summary of the variables used in a prototypical sandy beach is presented in Table 1.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydraulic conductivity, $K$</td>
<td>20 m/d</td>
</tr>
<tr>
<td>Mean tidal height, $D$</td>
<td>1.5 m</td>
</tr>
<tr>
<td>Soil porosity, $n_s$</td>
<td>0.25</td>
</tr>
<tr>
<td>Shallow water parameter, $e$</td>
<td>0.4</td>
</tr>
<tr>
<td>Mean tidal amplitude, $\alpha$</td>
<td>0.20 or 0.35</td>
</tr>
<tr>
<td>Coastline parameter, $\beta$</td>
<td>0.35 or various</td>
</tr>
</tbody>
</table>

Table 1: Input data for typical sandy beach condition.
Figure 2 shows the watertable fluctuation in terms of dimensionless distance in the $x$-direction for a complete tide cycle of $0 < T < 2\pi$. Between the sloping boundary and $X = 1$ in the $x$-direction, the watertable rises when the tide ebbs ($0 < T < \pi$) and decays when the tide rises ($\pi < T < 2\pi$).

Figures 3 and 4 are plotted to examine the watertable response to the tide's ebb and rise. The figures are also plotted in order to compare the linear and higher-order solutions. The linear solution is plotted for $H_{\omega_0}$, in which $\beta$, $\alpha'$ and $c = 0$ in Equation (18).

The watertable at the boundary is affected by the rise and ebb of the tide level. In Figure 3, the watertable at the boundary is higher in the early stage of the tide ebb. As the mean tide level reduces further, the watertable at the boundary decreases. Away from the boundary, the watertable propagates to the constant height where tidal influences are negligible.

In Figure 4, watertable fluctuations during tide rise are plotted. The increase of the mean tide level at the boundary increases the watertable height. Similar decay patterns can be observed for the watertable fluctuations during the tide rise. Between the boundary and $X = 2.5$, the tidal influence on watertable fluctuations is found to be significant. The plots for linear and higher-order solutions in Figure 3 and 4 show a relative difference about 2% at constant watertable heights between linear and higher-order solutions.

**Effects of a Variable Coastline**

Figure 5 shows the effect of rhythmic coastlines on watertable fluctuations. Recall that $\omega$ is defined as $kA_1$ in Equation (6), which, in terms of the ratio of coastline lengths and amplitudes, is $\omega_c A/L$. The common rhythmic coastline is often of greater wavelength ($L_c$) than the amplitude ($A_1$) ($L_c >> A_1$).

Effects of Beach Slopes

The solution derived in this paper considers the sloping beach boundary. Figures 7 to 10 show calculated watertable fluctuations for various beach slopes. The fluctuations increase inversely with beach slope. A steeper slope will result in a reduced influence on the watertable fluctuations. However, for a shallowly sloped beach, the influence on watertable fluctuations will be significant.
In this paper, a new two-dimensional analytical solution for tide-induced watertable fluctuations in a sloping beach is derived. In the new solution, the shallow water ($\epsilon$), amplitude ($\alpha$) and coastline ($\beta$) parameters are used in the perturbation expansion. The numerical results demonstrate that the higher-order components, beach slope and the coastline shape can markedly affect tide-induced groundwater fluctuations in coastal aquifer.

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LITERATURE CITED


