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The Shallow Water Waves Equation Based on the Moving Element Method with ADI Scheme

F. M. Pacheco Tena† and P. C. C. Rosman‡

†CTTMar
UNIVALI, Itajaí-SC
88303-202, Brazil
franklinmpt@univali.br

‡ PEnO
COPPE/UFRJ, Rio de Janeiro-RJ
21945-970, Brazil
pccr@peno.coppe.ufrj.br

ABSTRACT

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This work presents the development of an efficient scheme in the formulation of the shallow water wave equations, using the Moving Element Method (MEM). The MEM developed by SCUDELARI (1997) has the same flexibility of finite element method related to irregular boundaries and potentiality, the same computational performance of the finite difference method. In the space discretization of the shallow water equations was used the MEM, and in the time discretization was used an uncoupled scheme developed by ROSMAN (1997), that allows evaluating velocity components values explicitly and the surface elevation of the sea implicitly. This scheme was called Uncoupled MEM (UMEM), produces a significant reduction in the order and bandwidth, when compared to standard coupled systems. The solution of the system resultant was obtained with a direct method. An Alternate Implicit Direction (ADI) scheme was also used, with the purpose of reducing the computer execution time and to explore MEM's potentiality. The schemes consistency, efficiency and computational effort are verified through a comparison of the numerical results with the analytical results of the test problem of tidal propagation in quarter-annulus channel with quadratically varying bathymetry. The results show that schemes are efficient and accurate.

ADDITIONAL INDEX WORDS: *Hydrodynamic Model 2DH, Uncoupled Shallow Water Waves Equations.*

INTRODUCTION

Hydrodynamical models are important tools to predict circulation and pollutant transport in estuarine and coastal waters and their development and application have been increased in environmental engineering. Models are tools integrators of knowledge, without them are difficult to obtain a dynamic vision of processes in these complex environmental systems. The finite difference method (FDM) and the finite element method (FEM) are the numerical models most used in solving the two-dimensional depth-integrated (2DH) shallow water equations.

The FDM is simple to use for the algorithm's coding and very effective in dealing with non-linear advective terms in the shallow water waves; the FDM allows the use of schemes very efficient such as Alternating Direction Implicit (ADI) (BLUMBERG and MELLOR, 1987; CASSULLI, 1990). The traditional finite difference models for shallow water flow employ rectangular finite difference grids and a disadvantage of the method is the treatment of curved boundary; for a good representing of the curved boundary is necessary to increase the number of computer points. For overcome such difficulty the researches have introduced the boundary-fitted coordinates that allows a better degree of flexibility than regular grid (SHENG, 1984; LIN and CHANDLER_WILDE, 1996; and MUIN and SPAUDING, 1996).

The FEM has the advantage of its flexibility in modeling irregular coastlines and varying bathymetry (common in estuarine system). However when compared with the FDM it is more complex in the implementation of the algorithm and the computational cost is very expensive (LYNCH and GRAY, 1979; ROSMAN, 1987; FORTUNATO, 1996).

New researches show great efforts to get formulations with more accuracy or new strategy for the solution of equation's systems. News numerical methods are applied for solution of the shallow water waves equation, STELLING and VAN KESTER (1994) used the finite volume method (FVM) that allow to approach the differential equation through the balance of the conservation property involved (mass, momentum, etc.) at an elemental volume; HON, *et al*., (1999) used the multiquadratic method that does not require the generation of a grid, and the domain is composed of scattered collocation points. SCUDELARI (1997) presented the moving element method (MEM) that allow discretizing the domain with the same flexibility of MEF for irregular boundary and has potentially the same computational efficiency of the MDF.

The purpose of this paper is to present the development and testing of an uncoupled scheme for shallow water waves by using the MEM (UMEM). The paper first presents the governing equation and the uncoupled scheme developed by ROSMAN (1997) for shallow water waves; this scheme allows evaluating velocity components values explicitly and the surface elevation of the sea implicitly and produces a significant reduction in the order and bandwidth of the matrix system, when compared to standard coupled systems. The second step of the paper, presents a scheme ADI by using extrapolated functions in the matrix system of uncoupled scheme, the computational time of this scheme is very economical. The model is tested against analytic solutions tidal flow in an annular section channel with quadratic bottom topography..

MATHEMATICAL MODEL

The governing differential equations that describe the fluxes due to tides, can be obtained by the integration along the vertical of the three dimensional form of the shallow water equations. At region of study the flow is assumed to be vertically well mixed with a hydrostatic pressure gradient. The mathematical model is formed by the depth-integrated continuity and momentum equations in x and y directions

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}\left[U(h+\zeta)\right] + \frac{\partial}{\partial y}\left[V(h+\zeta)\right] = 0 \quad (1)$$

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -g\frac{\partial \zeta}{\partial x} + \frac{1}{\rho_0 H}\left(\frac{\partial (H\tau_{xx})}{\partial x} + \frac{\partial (H\tau_{xy})}{\partial y}\right) + fV + \frac{\tau_x^S}{\rho_0 H} - gU\frac{\sqrt{U^2+V^2}}{C^2_\beta H} \quad (2)$$

$$\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}=-g\frac{\partial \zeta}{\partial y}+\frac{1}{\rho_0 H}\left(\frac{\partial\left(H\tau_{xy}\right)}{\partial x}+\frac{\partial\left(H\tau_{yy}\right)}{\partial y}\right)-fU+\frac{\tau_y^S}{\rho_0 H}-gV\frac{\sqrt{U^2+V^2}}{C_\beta^2 H} \tag{3}$$

where t = time; U and V are the velocity components in the x and y directions respectively; h = mean water depth; ζ = sea surface elevation; $H = h+\zeta$ = total depth; ρ_0 = averaged water density; C = Chezy coefficient; f = Coriolis parameter; g = gravitational acceleration. The terms τ_{xx}, τ_{xy}, τ_{yy}, τ_{yx} represent the effects due to turbulent shear; and τ_x^s and τ_y^s are the wind stress components in the x and y directions respectively.

The governing equations (1)-(3) are subject to appropriate boundary and initial condition. The boundary conditions are zero flow normal to a solid boundary and specified water surface elevation or normal flow at on open boundary.

The solution of governing equations is made using an uncoupled scheme for shallow water, development by ROSMAN (1997). The scheme is based in the Implicit Factoring Method (ROSMAN, 1987) for the time discretization, allows writing explicitly velocities U and V in the momentum equations and writes implicitly the surface elevation ζ in the continuity equation. The substitution of the momentum equations into the continuity equation allows obtaining an equation in terms of surface elevation. The equation to solve is:

$$\left[\frac{2}{\Delta t}+\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}\right]\zeta^+ +\left[U-\frac{g}{2}\frac{\partial}{\partial x}\left(\frac{H}{C_x}\right)\right]\frac{\partial \zeta^+}{\partial x}-\frac{g}{2}\left(\frac{H}{C_x}\right)\frac{\partial^2\zeta^+}{\partial x^2}+\left[V-\frac{g}{2}\frac{\partial}{\partial y}\left(\frac{H}{C_y}\right)\right]\frac{\partial \zeta^+}{\partial y}-\frac{g}{2}\left(\frac{H}{C_y}\right)\frac{\partial^2\zeta^+}{\partial y^2}=\frac{2\zeta}{\Delta t}-\frac{\partial}{\partial x}\left(\frac{M_x}{C_x}H+Uh\right)-\frac{\partial}{\partial y}\left(\frac{M_y}{C_y}H+Vh\right) \tag{4}$$

and

$$U^+=\frac{1}{C_x}\left[-g\frac{1}{2}\frac{\partial\zeta^+}{\partial x}+M_x\right] \tag{5}$$

$$V^+=\frac{1}{C_y}\left[-g\frac{1}{2}\frac{\partial\zeta^+}{\partial y}+M_y\right] \tag{6}$$

the symbols (+) in ζ^+, U^+ and V^+ indicate the surface elevation and velocities in the time $t+\Delta t$. The values of the parameters C_x, C_y, M_x and M_y are known and we
re calculated by using the following expressions

$$C_x=\left[\frac{1}{\Delta t}+\frac{1}{2}\frac{\partial U}{\partial x}+\frac{1}{2}\beta^{\otimes}\right] \tag{7}$$

$$M_x=\left[\frac{U}{\Delta t}-\frac{1}{2}\left(U\frac{\partial U^\#}{\partial x}+V^\#\frac{\partial U}{\partial y}+V\frac{\partial U^\#}{\partial y}\right)-g\frac{1}{2}\frac{\partial\zeta}{\partial x}+\left\{\frac{1}{\rho_o H}\left(\frac{\partial\left(H\tau_{xx}\right)}{\partial x}+\frac{\partial\left(H\tau_{xy}\right)}{\partial y}+\tau_x^s\right)\right\}^{\otimes}+fV^{\otimes}-\frac{1}{2}\beta^{\otimes}U\right] \tag{8}$$

$$C_y=\left[\frac{1}{\Delta t}+\frac{1}{2}\frac{\partial V}{\partial x}+\frac{1}{2}\beta^{\otimes}\right] \tag{9}$$

$$M_y=\left[\frac{V}{\Delta t}-\frac{1}{2}\left(U^\#\frac{\partial V}{\partial x}+U\frac{\partial V^\#}{\partial x}+V\frac{\partial V^\#}{\partial y}\right)-g\frac{1}{2}\frac{\partial\zeta}{\partial y}+\left\{\frac{1}{\rho_o H}\left(\frac{\partial\left(H\tau_{xy}\right)}{\partial x}+\frac{\partial\left(H\tau_{yy}\right)}{\partial y}+\tau_y^s\right)\right\}^{\otimes}-fU^{\otimes}-\frac{1}{2}\beta^{\otimes}V\right] \tag{10}$$

where the symbol (#) indicates the extrapolated variable at the time $t+\Delta t$, and symbol ($^{\otimes}$) indicates the extrapolated variable at the time $t + {}^1/_2\,\Delta t$. To apply the equation (4) in all points of domain leads to obtain a linear system, this system has a matrix with non-symmetric band and the solution can be obtained with direct methods. In the spatial discretization of the equation (4) MEM is used and it will be described next.

MOVING ELEMENT METHOD

The mean idea of the MEM is that the domain (Ω) of the physical problem to be studied is discretized using points, on which will be defined elements of similar form as it has been used to the MEF (Ω^e). On these elements (they have overlapping domain) local functions of interpolation for each discrete point of the domain will be constructed. Figure 1 shows an example of the domain discretization; the dotted line indicates the domain of element 6 and the hatched line indicates the domain of element 11.

The approach of any function $u(x, y)$, situated in the position (x_P, y_P), that is located in the domain of an element (Ω^e) defined by NP points, can be made of the following form:

$$u(x_p,y_p)=\sum_{i=1}^{NP}u(x_i)\varphi_i(x_p,y_p) \tag{11}$$

$$\frac{\partial}{\partial x}[u(x_p,y_p)]=\sum_{i=1}^{NP}u(x_i)\frac{\partial}{\partial x}[\varphi_i(x_p,y_p)] \tag{12}$$

$$\frac{\partial^2}{\partial x^2}[u(x_p,y_p)]=\sum_{i=1}^{NP}u(x_i)\frac{\partial^2}{\partial x^2}[\varphi_i(x_p,y_p)] \tag{13}$$

where φ_i are the interpolation functions, that for the case of quadratic elements, they are the same ones that for the finite elements Lagrangian. PACHECO TENA (2001) worked φ_i with NP=9, and used a system of local coordinates (plain ξ- η) for calculating of $\varphi_i(x(\xi,\eta)$, $y(\xi,\eta))$ and their derivatives. The solution the uncoupled shallow water waves equation by using the moving element method was identified as UMEM model.

ADI SCHEME

To apply ADI scheme, the domain was divided in discrete points that are associated to a grid similar to used in finite differences with boundary-fitted as it shows in Figure 2. N_x vertical lines and N_y horizontal lines form this grid, so that the total number of points is N_x x N_y. By transforming the system obtained of the equation (4) into a tri-diagonal system was used

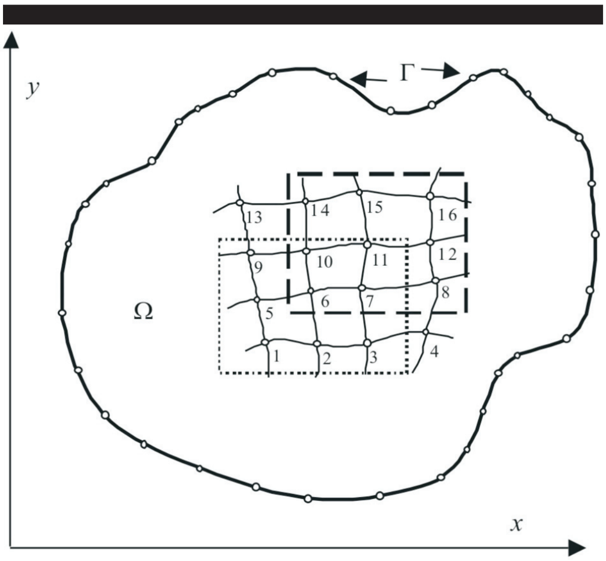

Figure 1. Domain discretization.

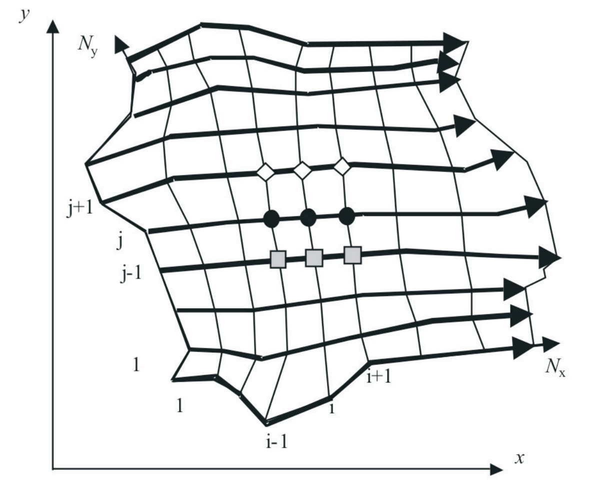

Figure 2. Grid for a basin of irregular boundary. The symbols (□), (◇) and (○) indicate the position of variables are known, variables are solved and variables extrapolated respectively.

quadratic extrapolation function for the time $t+\Delta t$,

$$U^{\#} = 3U - 3U^{-} + U^{=} \quad (14)$$

where the symbol (#) indicates the extrapolated variable at the time $t + \Delta t$; the symbols (-) and (=) indicate the variable at the times t - Δt and t - $2\Delta t$ respectively. The Figure 2 shows the implementing system tri-diagonal of the line j, the values of ζ^{+} in that line are unknown (O), the values of ζ^{+} in the line j-1 are known (□) and values of ζ^{+} in the line j+1 are known (◊) by using extrapolation functions, this one-dimensional problem is solved by direct method or double-sweep solution, as many times as necessaries, to reach the convergence. Computer execution time is reduced with ADI scheme.

Numerical Test and Discussions

The model formulation of UMEM and UMEM with ADI was tested by comparison with analytical solution for tidal forcing in an annular section channel. This is a standard test that has been used for many researchers; the challenge of the test is to get appropriate agreement between numerical and analytical results, and to demonstrate the control of the model to generate spurious oscillations of length $2\Delta x$. The analytical solution is presented in LYNCH and GRAY (1978).

The dimensions of the channel (Figure 3) are an inner radius R_1 = 60960m and outer radius R_2 = 152400m, the bathymetry varies quadratically from H_1 = 3.05m in R_1 until H_2 = 19.05m in R_2. The boundary conditions are along the land boundary the normal component of velocity is zero and at the open boundary a sinusoidal surface elevation is specified with period T = 12h, amplitude a = 0.03048m, the bottom friction parameter $\beta=10^{-4}$ s^{-1} and with several time step Δt = 1800s, Δt = 900s and Δt = 450s. The physical domain is discretized with a grid that has 221 points.

In Figures 4-5, is shown the spatial distribution of surface elevation and radial velocity for different phases of the tide (0h, 1.5h, 3h, 4.5h, 6h, 7.5h, 9h, 10.5h and 12h) so for UMEM and UMEM_ADI models. The response of the numerical models are represented the symbols ◇ = maximum values and + = minimum values. The lines represent the analytical solutions. The numerical oscillations can be observed, when the symbols diamond and cross are not equal. The response of the UMEM model (Figure 4) shows that it has very small numerical oscillations so for ζ and V_R (radial velocity). The response of the UMEM_ADI model (Figure 5) shows it has some numerical oscillations near of R_1 and R_2.

For quantifying the amplitude of the oscillations is defined a

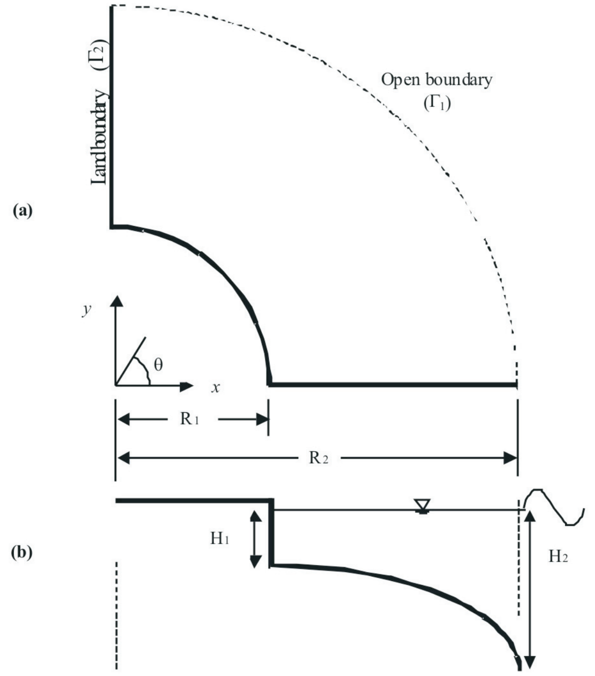

Figure 3. Geometry of the Annular Section Channel .

error relative (δ) with respect to analytic solution. For ζ, it is given for

$$\delta^{\zeta} = \frac{\zeta_M - \zeta_T}{\left|\zeta_{\max T}\right|} \times 100\% \quad (15)$$

where ζ_M = average of numerical values in a radial position, ζ_T = analytic value and $\zeta_{\max T}$ =maximum value analytic of a tide period at a radial position.

The relative errors for ζ and V_R are calculated for several radial distances, from R_1 to R_2 with ΔR = 7620m. With the UMEM model elevation errors of (δ^{ζ}) less than 1.5% were found, with Δt = 1800s and this happen near of R_1. The δ^{ζ} values are diminished when the Δt is decreased; for Δt = 450s the δ^{ζ} are less than 0.5%. The radial velocity errors were found in a band from ± 2.5% for Δt = 1800s, to ± 1.5% for Δt = 450s. It was observed in R_1 the reduction of the errors when the time step diminishes.

The relative errors of the free surface elevation calculated with MEMD_ADI, were found in a band of ± 2.5%, remaining itself in this band for the different Δt used in the simulation. The errors of the radial velocities , were found in a bigger band; that is ± 4,0%. The biggest errors are observed in R_1 (7,5%), which diminish significantly when the Δt of the simulation diminishes (4%). The velocity errors calculated in R_1 were neglected because the theoretical velocity in this region is null, causing an indeterminate value in R_1 in equation (15).

The performance of computational time of MEMD and MEMD_ADI was investigated through the comparison with the model FIST (Fist in Space and Time), developed by ROSMAN (1987); this model resolves the three governing equations of the shallow water waves (equations 1, 2 and 3), using in the space discretization the finite elements method. It was observed that the UMEM is approximately 21.7 times faster than the FIST. For the same problem, the UMEM_ADI revealed 2.6 faster than the UMEM, and 56.7 times faster than the FIST.

CONCLUSIONS

In the present work a numerical model of circulation hydrodynamics 2DH was presented, it is based an uncoupled scheme for shallow water waves and the moving element

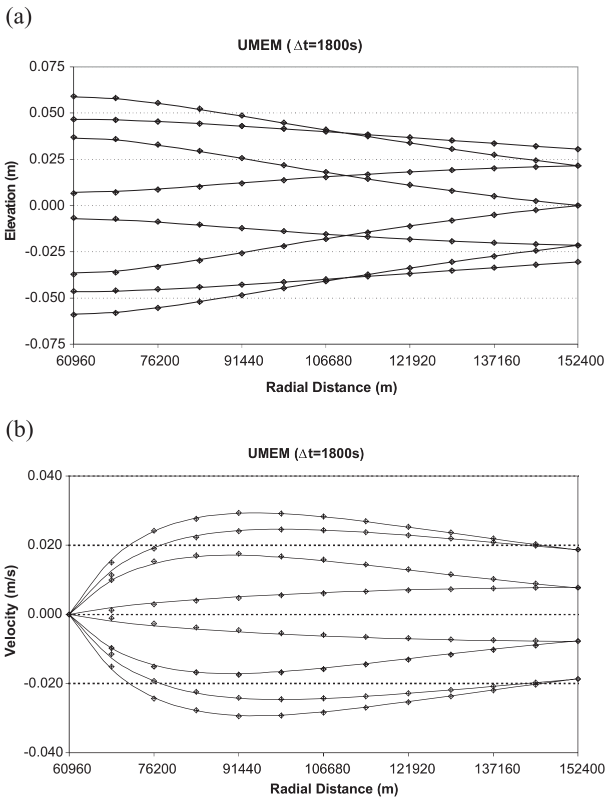

Figure 4. UMEM: Maximum and minimum numerical results (symbols) for surface elevation (a) and radial velocity (b), and analytical solution (lines) for different phases of tide (0h, 1.5h, 3h, 4.5h, 6h, 7.5h, 9h, 10.5h and 12h).

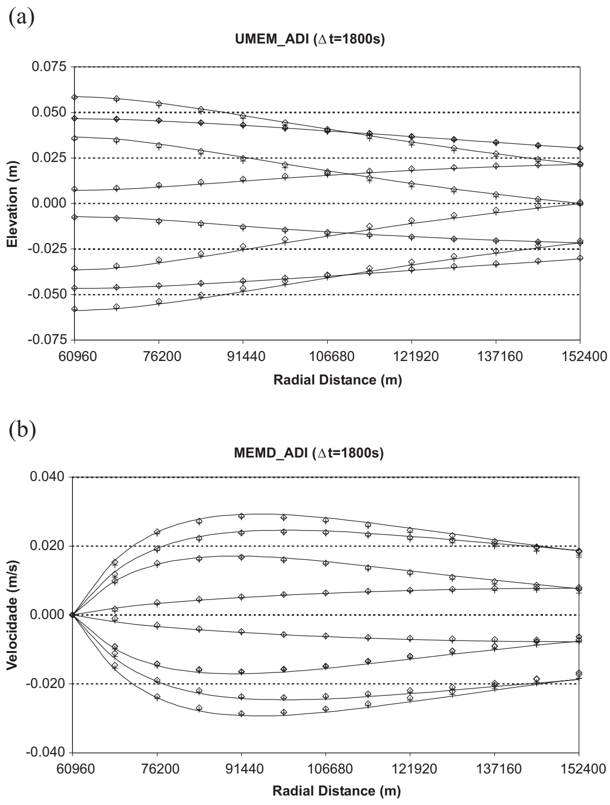

Figure 5. UMEM_ADI: Maximum and minimum numerical results (symbols) for surface elevation and radial velocity, and analytical solution (lines) for different phases of tide (0h, 1.5h, 3h, 4.5h, 6h, 7.5h, 9h, 10.5h and 12h).

method (UMEM), this allows to reduce the size of the matrix of the resultant system, as much in the order as in the width of the band, providing economy of time of processing in the solution of the matrix. It was developed a scheme ADI together UMEM (UMEM_ADI), which allows transforming the system of equations in a set of small systems tri-diagonal. The UMEM and UMEM_ADI models reveal that they are efficient, accuracy and steady, in the solution of circulation for tidal forcing in an annular section channel.

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